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The noise from the large-scale structure of a jet

By J. E. FFOWCS WILLIAMS AND A. J. KEMPTON†
Engineering Department, University of Cambridge

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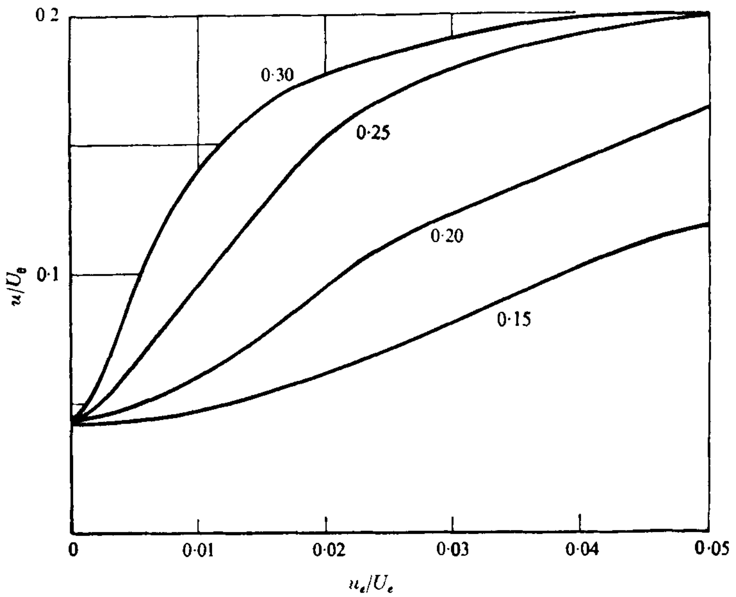
In this paper we assess the importance as a noise source of the well-ordered large-scale structure of a jet. We propose two simple models of the structure: the first emphasizes those features in common with waves that initially grow on an unstable shear layer but eventually saturate and decay, while the second regards the abrupt pairing of eddies as the most significant event in the jet's development. Our models demonstrate the possibility that forcing at one frequency could increase the *broad-band* noise of a jet, though, for jets with supersonic eddy convection velocities, the sound propagating in the direction of the Mach angle retains the spectrum of the excitation field. These features are consistent with the available experimental data, and strongly support the view that the large-scale structure of jet turbulence provides the dominant contribution to jet noise.

1. Introduction

More and more experiments are suggesting that beneath the chaos of a turbulent jet there exists a large-scale structure which is quite well ordered even at high Reynolds numbers (Mollø-Christensen 1967; Lau, Fuchs & Fisher 1972; Laufer, Kaplan & Chu 1973; Moore 1977). Sophisticated methods are often used to extract this coherent structure from the background turbulence. Several workers (Petersen, Kaplan & Laufer 1974; Lau & Fisher 1975; Moore 1977) have used 'eduction' techniques. They chose a suitable trigger condition that marked a definite stage in the passage of an organized large-scale structure, and then averaged recordings of the flow at a fixed position and at a known time interval before or after it. Others (Crow & Champagne 1971; Moore 1977) have forced the jet at the frequency of a disturbance that would grow naturally and have raised the latent structure above the level of the background turbulence; the coherent fluctuations are amplified and become much more steady and measurable. (It is likely that the structure of the forced jet is qualitatively similar to that of a jet allowed to develop naturally.) Finally, much information can be deduced from experiments with flows at lower Reynolds number (Brown & Roshko 1974; Winant & Browand 1974), since the large-scale structure is believed to be relatively insensitive to changes in Reynolds number.

Crow & Champagne (1971) used a loudspeaker upstream of the nozzle to impose a periodic surging of controllable frequency and amplitude at the nozzle exit, and studied the response downstream with hot-wire anemometry and schlieren photography. They discovered that surging amplifies the corresponding instability wave. The wave grows in accordance with linear stability theory in the initial stages of the

† Present address: Noise Department, Rolls-Royce Limited, P.O. Box 30, Derby, England.



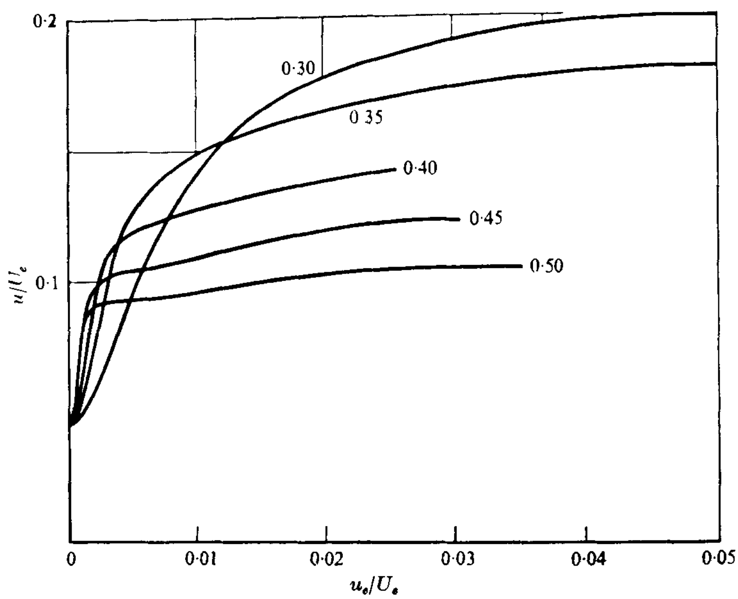


FIGURE 1. Response of the jet to different levels of forcing (Crow & Champagne 1971). (*a*) Amplitude-response functions measured on the centre-line four diameters downstream of the jet exit. The response functions are labelled with Strouhal numbers, which range from 0·15 to 0·30. (*b*) Amplitude-response functions continued through the Strouhal-number interval 0·30–0·50.

jet's development, but of course it does not grow indefinitely; a finite limit is attained. This saturation limit varies for different surging frequencies and is greatest for a Strouhal number of about 0·3 based on the jet velocity and nozzle diameter (figure 1). The response of the jet at different locations is illustrated in figure 2. Initially the waves grow very quickly, but the growth rate gradually reduces downstream and

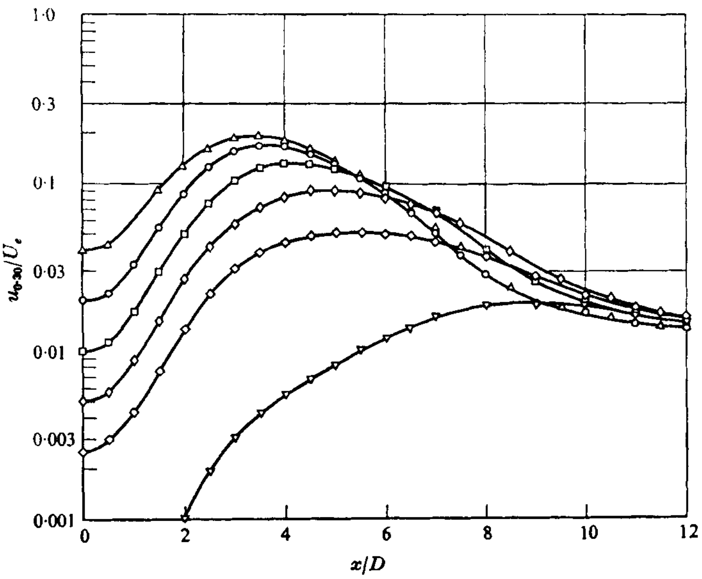


FIGURE 2. Response of a jet at different locations (Crow & Champagne 1971). Centre-line intensity profiles of the fundamental wave driven at a Strouhal number of 0·30. Forcing amplitude u_e/U_e: ▽, no forcing; ◇, 0·25% forcing; ◇, 0·5%; □, 1%; ○, 2%; △, 4%. The ordinate $u_{0·30}/U_e$ is logarithmic, so the forced profiles would have had the same shape had the jet been linear.

eventually the waves decay as the velocity profile becomes more and more stable, and as small-scale turbulence 'dissipates' the coherent motion. As the level of surging is increased, the transition from growth to decay occurs earlier, more abruptly and initially with a larger value for the maximum response.

Liu (1974) and Chan (1974*a*, *b*) have modelled these large-scale instabilities by splitting the total flow into three components: the time-independent mean flow, the instability wave and the turbulent fluctuations. The nonlinear development of waves was represented as a product of an amplitude function, determined by a balance between the energy transferred from the mean flow and that transferred to the turbulent fluctuations, and a shape function, determined from linear stability analysis for the local mean flow. Crighton & Gaster (1976) adopted a different approach and modelled the jet's development with linear instability waves on a laminar shear layer whose mean velocity is 'slowly varying'. Their analysis avoided the 'closure' assumptions required by Liu and Chan, but was restricted to linear waves. Both the theory of Chan and the theory of Crighton & Gaster predict well the growth of the waves and the approach to peak amplitude, but good agreement with experiment is not so easy to obtain in the decaying portion of the flow.

It is surprising that instability waves growing and decaying on a shear layer account for some aspects of the large-scale structure of turbulent jets as well as they do since the eddies are known to coalesce and wave crests are 'lost'. 'The new "wave length" some distance downstream is not due to the slow change of wave number and frequency, but to an actual merging of two local maxima' (Laufer *et al.* 1973).

Experiments on flows at lower Reynolds numbers (Brown & Roshko 1974; Winant & Browand 1974) show that waves do grow in the initial stages of the jet's development,

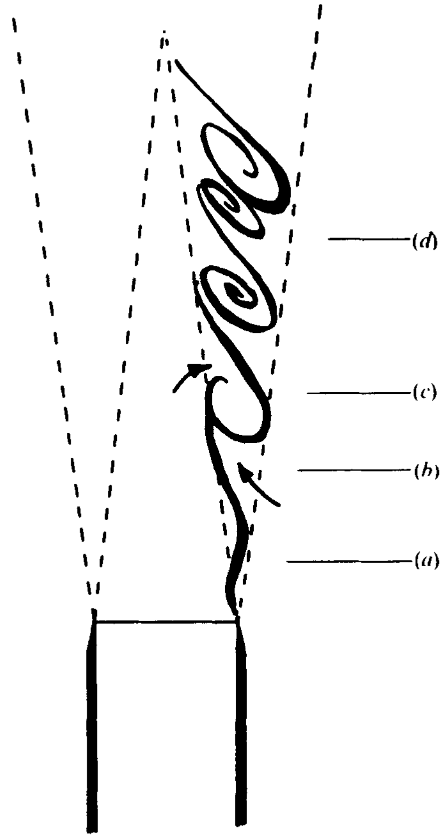


FIGURE 3. Schematic diagram of development of a jet shear layer (Moore 1977). (*a*) Shear layer oscillates. (*b*) Air becomes entrained. (*c*) Vortices form. (*d*) Vortices form pairs and so increase axial spacing.

but after two or three wavelengths they steepen and roll up into discrete concentrations of vorticity. These eddies convect downstream and interact with their neighbours by rolling around each other, deforming and coalescing (figure 3). Viscous diffusion then smears out the identities of the individual vortices to leave a single larger vortex. The disappearance of eddies can also occur when two strong vortices drain or tear apart a weak neighbour (Damms & Küchemann 1974; Moore & Saffmann 1975). The pairing of eddies arises because of small variations in their strengths and spacings, and does not always occur at the same point in space. In fact the positions of pairings must vary over a distance comparable to the eddy separation if the shear-layer spread is to be smooth and linear (Petersen *et al.* 1974).

Numerical modellings of two-dimensional shear layers and axisymmetric jet flows by Acton (1976) predict many of the features of the large-scale structures found experimentally. Eddies evolve and coalesce as they convect downstream, and the coalescence always occurs abruptly, being provoked by lateral or radial irregularities in the shear layer. Two stages in the development of an unforced axisymmetric jet obtained by Acton are illustrated in figure 4. The radial velocity has a maximum just downstream and a minimum just upstream of the eddies, and its axial distribution

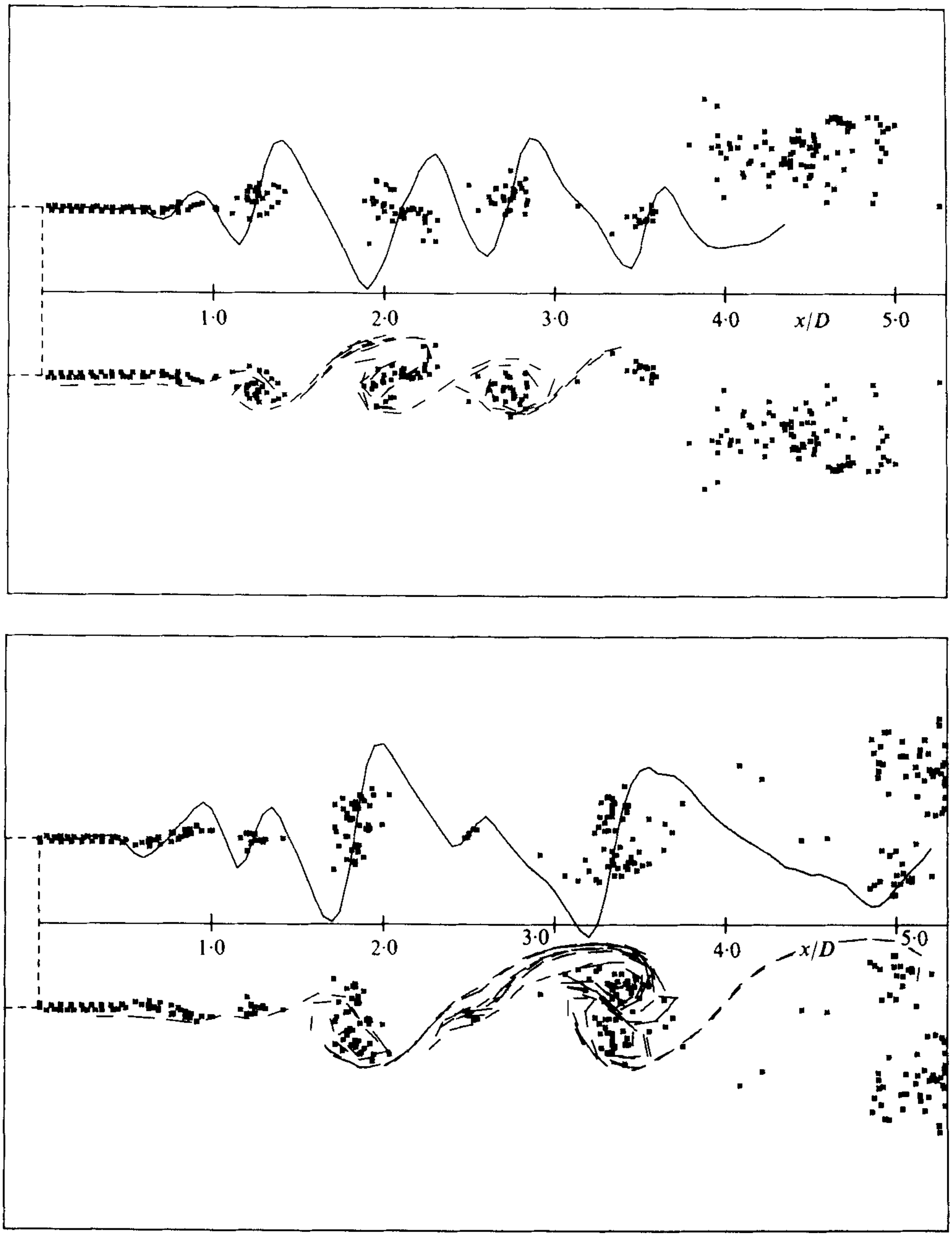


FIGURE 4. Two stages in the development of an unforced axisymmetric jet (Acton 1976). ——, radial velocity trace; – – –, dye line; ■, vortex element.

provides a concise description of the large-scale structure of the jet. A composite picture of the jet's development obtained from the radial velocity traces at successive times is illustrated in figure 5.

Experiments show that forcing increases the strength of the waves and eddies, and makes the spacings more regular. It induces those unstable waves at the forcing

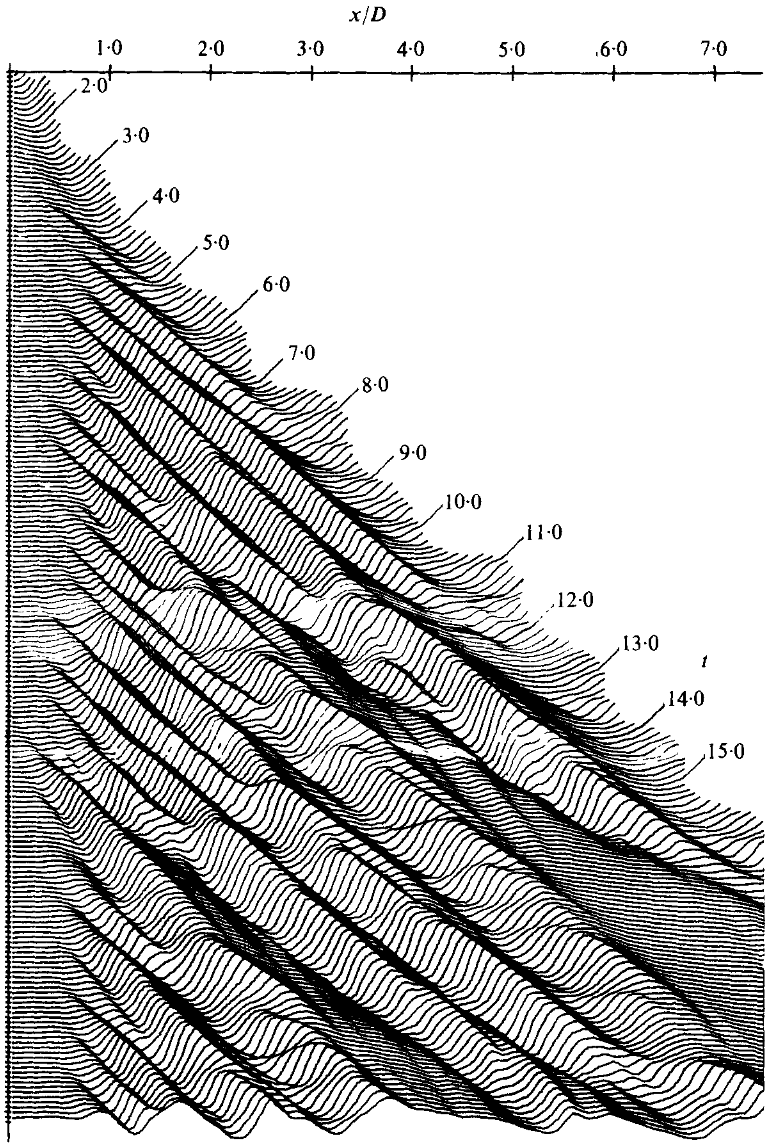


FIGURE 5. Axial distribution of the radial velocity at consecutive times (Acton 1976). Unforced jet.

frequency to grow and form the dominant eddies. A composite picture obtained by Acton of a jet forced at a Strouhal number of 1·5 is illustrated in figure 6.

As well as increasing the strength and regularity of the jet's coherent structures, forcing by a loudspeaker upstream of the nozzle exit modifies the radiated sound. Indeed Crow (1972) claimed that the jet could act as an amplifier of an internal tone and that a gain of 34 dB was possible. Moore (1977), however, found no evidence of the amplification of internal tones; instead high levels of forcing increased by up to 7 dB the radiated *broad-band* noise (see also Bechert & Pfizenmaier 1975). Moore demonstrated that exciting the shear layer at certain frequencies with a fluctuating pressure

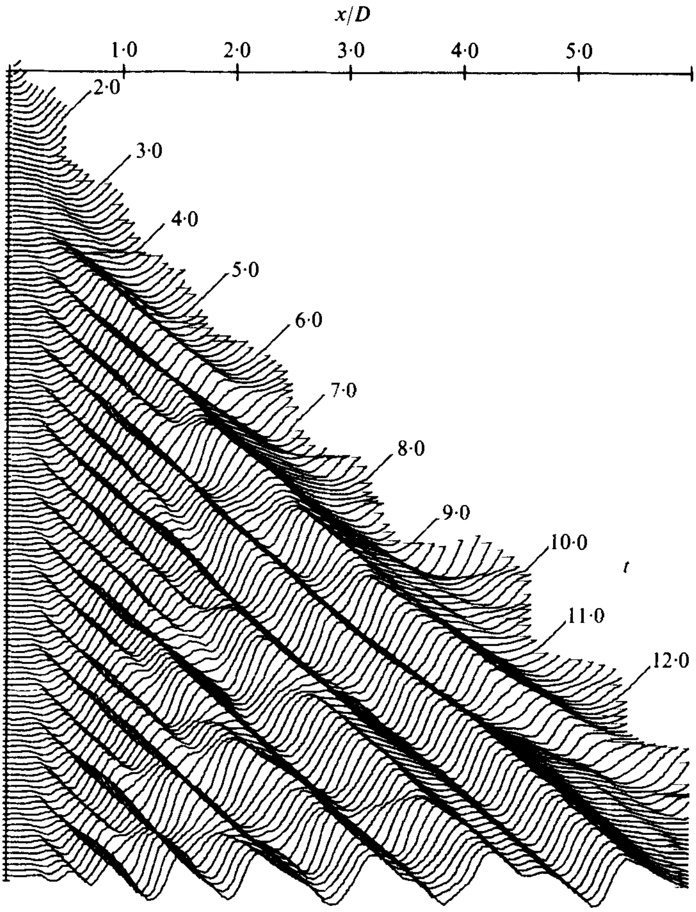


FIGURE 6. Axial distribution of the radial velocity at consecutive times (Acton 1976). Forced jet, $St = 1{\cdot}5$.

of only 0·08 % of the jet dynamic head increases the noise over the whole subsonic Mach number range. He found that the field shapes for the sound from the excited and unexcited jet were very similar; the spectra are also similar, but for the excited jet the peaks are steeper and the peak frequency varies with the excitation frequency and not with the flow speed. Unpublished work by Moore shows that regardless of the frequency *all the extra broad-band noise from a forced jet orginates at the same place,* about three diameters downstream of the nozzle exit. This is unlike the noise of an unexcited jet, where high frequencies come from near the nozzle exit and lower frequencies from further downstream. Moore (1977) also found that forcing the jet at high frequencies can *reduce* the jet's broad-band noise when the boundary layer inside the nozzle is thick.

The relationship between the large-scale structure in a jet and the radiated noise has been partly analysed by Crighton (1972). He modelled the interaction of the large-scale structure with the nozzle lip, and argued that a strong source of sound is associated with eddy-induced variations in the jet mass flow and thrust. Since upstream forcing can increase the strength of the eddies, the jet could act as an amplifier of internal noise, drawing energy from the mean flow. The upstream noise source triggers instabilities of the shear layer which react back on the nozzle to generate more noise, but this noise is always at the frequency of the internal source.

The coherent large-scale structure will also radiate sound efficiently by itself since the motion that generates the noise is abrupt and correlated over large distances. In this paper we attempt to assess how significantly it contributes to the radiated sound. We propose models of the source structure which we believe simulate nonlinear features of the jet structure that have important acoustic consequences. These centre on the change from growth to decay of instability waves and on the coalescence of vortex rings. But the source structures are not buttressed by any nonlinear theory. Indeed all the mathematics we use is linear. We are looking only for general trends and have therefore adopted a fairly heuristic approach. But, despite the many obvious deficiencies of our models, we have been able to draw from them firm conclusions concerning the radiated sound and many of these conclusions accord with experimental fact.

We first model (in § 2) the source structure as a travelling instability wave that grows and decays in amplitude. Its phase velocity has a small random component, so that the phase of the wave varies at the 'break-point' where the change from growth to decay occurs. If this random variation is small, we find that the radiated sound has the same frequency as the instability wave, but if the variation is large *broad-band* noise results.

We next model (in § 5) the coalescence of eddies in the jet. Powell (1964) has shown that this pairing radiates sound with the quadrupole characteristics of jet noise, and Laufer *et al.* (1973) and Winant & Browand (1974) have suggested that it is the mechanism primarily responsible for the generation of jet noise. We model the jet as a series of eddies that convect downstream at a constant subsonic speed until coalescence occurs. When they reach the position of coalescence, the eddies merge in pairs and then continue convecting downstream at the same constant speed. We do not model the details of the pairing, but we do not believe that these details are essential. We assume that the position of coalescence varies randomly, and find that if it varies over a distance comparable to the eddy spacing broad-band noise is radiated.

This vortex-pairing model of the large-scale structure is inappropriate for jets if the convection velocity is greater than the ambient speed of sound. For at the Mach angle an eddy pattern is noisy even if it is 'frozen'. To model such a jet realistically we must take account of the finite lifetime of the eddies, and this we do (in § 8) by again assuming that the sources have the structure of a travelling wave that grows and decays in amplitude; but now we assume that the position of the maximum amplitude fluctuates. We find that at the Mach angle sound is radiated only at the wave's frequency, even if the position of 'breaking' varies a lot.

2. A wave model

Crow (1972) (see also Crighton 1975) proposed a 'wave-antenna' model of the large-scale structure of the jet and estimated the sound it would radiate. It is an extension of his model that we first describe. We assume, as Crow did, that the fluctuations vary sinusoidally along the jet, and that their amplitude at first grows and then decays as they are convected downstream of the nozzle at $x = -x_0$: the wave strength is proportional to

$$\cos\{\omega_0 t - \omega_0(x+x_0)/U_0\}\exp\{-x^2/l^2\} \tag{1}$$

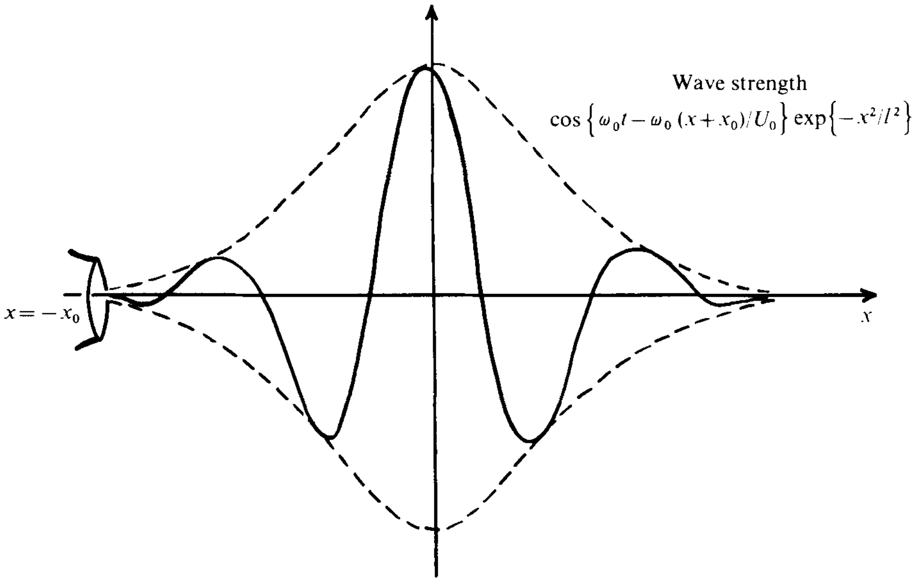


FIGURE 7. A wave model of the large-scale structure of a jet (Crow 1972).

(see figure 7). Here U_0 is the eddy convection velocity, ω_0 the radian frequency, and l the length scale over which the amplitude of the instability wave changes significantly. We require the wave amplitude to be small at the nozzle exit so that $x_0/l \gg 1$.

We extend this model to include an element of randomness. Because of random fluctuations in the flow, the response of the instability wave to forcing will vary in time (the phase and the amplitude at the nozzle exit will fluctuate). In addition, the randomness in the jet flow velocity profile will cause the phase velocity of the wave to vary. We shall show that, if these variations result in large random fluctuations in the phase of the instability wave at the 'break-point' where transition from growth to decay occurs, then broad-band noise is radiated. If they do not, the frequency of the radiated sound is the same as the frequency of the instability wave.

We determine the effect of randomness in the phase velocity by assuming that the fluctuating wave strength is proportional to

$$\cos\{\omega_0 t - \omega_0(x+x_0)[1+\epsilon(t-x/U_0)]/U_0\}\exp\{-x^2/l^2\}, \tag{2}$$

where ϵ is a small normally distributed random function with zero mean. The randomness in the phase grows as the wave convects downstream, although the development of each wave packet is deterministic (its phase velocity remains fixed).

3. The sound radiated by an instability wave

We obtain an estimate of the sound radiated from a turbulent jet by solving for the quadrupole-driven density field, given by

$$\frac{\partial^2\rho}{\partial t^2} - c^2\frac{\partial^2\rho}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i\,\partial x_j}, \tag{3}$$

with Lighthill's (1952, 1954) stress tensor estimated as

$$T_{ij} = \rho_0\,\tilde{u}^2 D^2 \delta_{ij}\,\delta(y)\,\delta(z)\cos\{\omega_0 t - \omega_0(x+x_0)[1+\epsilon(t-x/U_0)]/U_0\}\exp\{-x^2/l^2\}. \tag{4}$$

Here D is the jet diameter, c the speed of sound, ρ_0 the ambient density and $\tilde{u}$ a measure of the magnitude of the velocity fluctuations attained. We have represented the jet as an infinite line source on the jet axis with a structure similar to the wave structure (2). The source strength varies sinusoidally along the jet with an amplitude which at first grows and then decays, but as it convects downstream the sinusoidal pattern gradually loses its coherence. We have deliberately neglected the interaction of the instability wave with the nozzle lip, for provided that $x_0 \gg l$ the wave will be exponentially small at the nozzle exit. We can then also assume that the line source extends from $-\infty$ to $+\infty$.

The far-field fluctuating pressure is

$$\lim_{R\to\infty} p(\mathbf{x},t) = \frac{\rho_0 \tilde{u}^2 D^2}{4\pi c^2 R}\frac{\partial^2}{\partial t^2}\left\{\int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{l^2}\right)\cos\left(\omega_0 t - \frac{\omega_0 R}{c} + \frac{\omega_0 \xi\cos\theta}{c} - \frac{\omega_0(\xi+x_0)}{U_0}(1+\epsilon\{t-R/c+\xi(1-M\cos\theta)/U_0\})\right)d\xi\right\}, \quad (5)$$

where $R = |\mathbf{x}|$, $\cos\theta = x/R$ and $M = U_0/c$. The constant-bandwidth spectrum of the radiated sound is defined by

$$I(\omega) = \lim_{R\to\infty}\int_{-\infty}^{\infty}\frac{\langle p(\mathbf{x},t)\,p(\mathbf{x},t+\tau)\rangle}{\rho_0 c}\cos\omega\tau\,d\tau, \quad (6)$$

where the angle brackets denote the expectation value. So

$$I(\omega) = \frac{\rho_0\tilde{u}^4 D^4}{16\pi^2 c^5 R^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\omega\tau\exp\left(-\frac{\xi^2}{l^2}-\frac{\zeta^2}{l^2}\right)$$
$$\times\left\langle\frac{\partial^2}{\partial t^2}\cos\left(\omega_0 t-\frac{\omega_0 R}{c}+\frac{\omega_0\xi\cos\theta}{c}-\frac{\omega_0(\xi+x_0)}{U_0}\{1+\epsilon(t-R/c-\xi(1-M\cos\theta)/U_0)\}\right)\right.$$
$$\times\frac{\partial^2}{\partial t^2}\cos\left(\omega_0 t+\omega_0\tau-\frac{\omega_0 R}{c}+\frac{\omega_0\zeta\cos\theta}{c}\right.$$
$$\left.\left.-\frac{\omega_0(\zeta+x_0)}{c}\{1+\epsilon(t+\tau-R/c-\zeta(1-M\cos\theta)/U_0)\}\right)\right\rangle d\xi\,d\zeta\,d\tau. \quad (7)$$

A normally distributed random function, $\gamma(t)$ say, with zero mean has the property that all its autocorrelation functions except that of second order vanish. The autocorrelation functions are defined by

$$k_n(t_1,t_2,\ldots,t_n) = \langle\gamma(t_1)\,\gamma(t_2)\ldots\gamma(t_n)\rangle.$$

It follows (see Stratanovitch 1963, §3) that for any real α_1 and α_2

$$\langle\exp\{i\alpha_1\gamma(t_1)+i\alpha_2\gamma(t_2)\}\rangle = \exp\{-\tfrac{1}{2}\langle\gamma^2\rangle(\alpha_1^2+2R_\gamma(t_1-t_2)\,\alpha_1\alpha_2+\alpha_2^2)\}, \quad (8)$$

where R_γ is the autocorrelation coefficient, i.e.

$$\langle\gamma(t_1)\,\gamma(t_2)\rangle = \langle\gamma^2\rangle R_\gamma(t_1-t_2).$$

So after some manipulation we find that

$$I(\omega) = \frac{\rho_0\tilde{u}^4\omega^4 D^4}{32\pi^2 c^5 R^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\omega\tau\cos\{\omega_0\tau+\omega_0(\xi-\zeta)(1-M\cos\theta)/U_0\}$$
$$\times\exp\left(-\frac{\xi^2}{l^2}-\frac{\zeta^2}{l^2}-\frac{\omega_0^2\langle\epsilon^2\rangle}{2U_0^2}\{(\xi+x_0)^2+(\zeta+x_0)^2\right.$$
$$\left.-2(\xi+x_0)(\zeta+x_0)R_\epsilon(\tau+(\xi-\zeta(1-M\cos\theta)/U_0)\}\right)d\xi\,d\zeta\,d\tau, \quad (9)$$

where R_ϵ is the autocorrelation coefficient for the random function ϵ.

In the limit when the randomness in the phase at the break-point is small, i.e. $\omega_0^2\langle\epsilon^2\rangle x_0^2/U_0^2 \ll 1$, the radiated sound is strongly peaked at the frequency of the instability wave. For, if we expand the integrand as a power series in $\omega_0^2\langle\epsilon^2\rangle x_0^2/U_0^2$, we find to lowest order that

$$I(\omega) = \frac{\rho_0 \tilde{u}^4 \omega^4 D^4 l^2}{32c^5R^2} \exp\{-\tfrac{1}{2}k_0^2 l^2(1-M\cos\theta)^2\}\{\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\}, \tag{10}$$

where $k_0 = \omega_0/U_0$. (This result is essentially the same as that obtained by Crow 1972.) If the growth of the instability wave is slow on a wavelength scale, i.e. $k_0 l \gg 1$, the radiated sound is exponentially small away from the Mach angle.

At the other extreme when the randomness is large, i.e. $\omega_0^2\langle\epsilon^2\rangle x_0^2/U_0^2 \gg 1$, broad-band noise is radiated. In (9) the integration with respect to τ can be performed by the method of steepest descents, the integral being dominated by the contribution from the region near $\tau + (\xi-\zeta)(1-M\cos\theta)/U_0 = 0$. Thus

$$I(\omega) \simeq \frac{\rho_0 \tilde{u}^4 \omega^4 D^4}{32\pi^2 c^5 R^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\{-\omega(\xi-\zeta)(1-M\cos\theta)/U_0\}$$
$$\times\left(\frac{\pi T_\epsilon^2 U_0^2}{\omega_0^2\langle\epsilon^2\rangle(\xi+x_0)(\zeta+x_0)}\right)^{\frac{1}{2}} \exp\left(-\frac{\xi^2}{l^2}-\frac{\zeta^2}{l^2}-\frac{\omega_0^2\langle\epsilon^2\rangle(\xi-\zeta)^2}{2U_0^2}\right)d\xi\, d\zeta, \tag{11}$$

where T_ϵ is a measure of the time over which ϵ is well correlated;

$$d^2R_\epsilon(\tau)/d\tau^2 = -2/T_\epsilon^2 \quad \text{when} \quad \tau = 0.$$

We expand the square root in powers of ξ/x_0 and ζ/x_0 to find with a relative error of order l^2/x_0^2 that

$$I(\omega) \simeq \frac{\rho_0 \tilde{u}^4 \omega^4 D^4 l^2}{32\pi^{\frac{1}{2}} c^5 R^2}\,\frac{U_0}{\omega_0\langle\epsilon^2\rangle^{\frac{1}{2}} x_0}\left(1+\frac{\omega_0^2\langle\epsilon^2\rangle l^2}{U_0^2}\right)^{-\frac{1}{2}}$$
$$\times T_\epsilon \exp\left[-\tfrac{1}{2}k^2 l^2(1-M\cos\theta)^2\left(1+\frac{\omega_0^2\langle\epsilon^2\rangle l^2}{U_0^2}\right)^{-1}\right], \tag{12}$$

where $k = \omega/U_0$. Noise is radiated over a wide frequency range. If the growth and decay of the amplitude of the wave is rapid compared with the growth of the randomness in its phase, then $\omega_0^2\langle\epsilon^2\rangle l^2/U_0^2 \ll 1$.

An expression for the radiated sound that patches onto the solutions for both large and small randomness as $\langle\epsilon^2\rangle x_0^2/U_0^2 T_\epsilon^2$ tends to infinity and zero respectively can be found by approximating $R_\epsilon(\tau)$ by $1-\tau^2/T_\epsilon^2$ in (9). (This approximation is strictly valid only for large randomness $\omega_0^2\langle\epsilon^2\rangle x_0^2/U_0^2 \gg 1$.) The analysis is the same as that used in the method of steepest descents except that the phase terms are not approximated by their values at the stationary point $\tau = -(\xi-\zeta)(1-M\cos\theta)/U_0$. So

$$I(\omega) \simeq \frac{\rho_0 \tilde{u}^4 \omega^4 D^4}{32\pi^2 c^5 R^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\omega\tau\cos\{\omega_0\tau+\omega_0(\xi-\zeta)(1-M\cos\theta)/U_0\}$$
$$\times\exp\left(-\frac{\xi^2}{l^2}-\frac{\zeta^2}{l^2}-\frac{\omega_0^2\langle\epsilon^2\rangle(\xi-\zeta)^2}{2U_0^2}-\frac{\omega_0^2\langle\epsilon^2\rangle(\xi+x_0)(\zeta+x_0)}{U_0^2T_\epsilon^2}\right.$$
$$\left.\times\{\tau+(\xi-\zeta)(1-M\cos\theta)/U_0\}^2\right)d\xi\, d\zeta\, d\tau, \tag{13}$$

and after some manipulation we find

$$I(\omega) \simeq \frac{\rho_0 \tilde{u}^4 \omega^4 D^4 l^2}{32\pi^{\frac{1}{2}} c^5 R^2} \frac{U_0}{\omega_0 \langle \epsilon^2 \rangle^{\frac{1}{2}} x_0} \left(1 + \frac{\omega_0^2 \langle \epsilon^2 \rangle l^2}{U_0^2}\right)^{-\frac{1}{2}}$$
$$\times T_\epsilon \exp\left[-\tfrac{1}{2} k^2 l^2 (1 - M\cos\theta)^2 \left(1 + \frac{\omega_0^2 \langle \epsilon^2 \rangle l^2}{U_0^2}\right)^{-1}\right]$$
$$\times \frac{1}{2}\left[\exp\left(-\frac{(\omega+\omega_0)^2 U_0^2 T_\epsilon^2}{4\omega_0^2 \langle \epsilon^2 \rangle x_0^2}\right) + \exp\left(-\frac{(\omega-\omega_0)^2 U_0^2 T_\epsilon^2}{4\omega_0^2 \langle \epsilon^2 \rangle x_0^2}\right)\right], \tag{14}$$

with a relative error of order l^2/x_0^2. Because its frequency has a random modulation, the instability wave can be decomposed into a broad spectrum of waves and each of these waves radiates noise at its own frequency. The width of the spectrum is proportional to $\omega_0 \langle \epsilon^2 \rangle^{\frac{1}{2}} x_0 / U_0$, the randomness in the phase of the wave at the break-point, and inversely proportional to T_ϵ, the correlation time for the randomness.

To obtain the overall sound intensity I, we integrate the power spectrum over all frequencies. For weak forcing the intensity is simply

$$I \simeq \frac{\rho_0 \tilde{u}^4 \omega_0^4 D^4 l^2}{16 c^5 R^2} \exp\{-\tfrac{1}{2} k_0^2 l^2 (1 - M\cos\theta)^2\}. \tag{15}$$

For strong forcing, when the power spectrum is given by (12), the intensity is

$$I \simeq \tfrac{3}{32} 2^{\frac{1}{2}} \frac{\rho_0 \tilde{u}^4 U_0^4 D^4}{c^5 R^2 l^2} \frac{1}{k_0 l} \frac{U_0 T_\epsilon}{\langle \epsilon^2 \rangle^{\frac{1}{2}} x_0} \left(1 + \frac{\omega_0^2 \langle \epsilon^2 \rangle l^2}{U_0^2}\right)^2 (1 - M\cos\theta)^{-5}. \tag{16}$$

The intensity is amplified by five inverse powers of the Doppler factor because of source convection, but the singularity at the Mach angle is a spurious result; it is a feature of the method of steepest descents, which neglects variations in the phase. We cannot neglect such variations when the frequency of the radiated sound is large, but must use the expression for the power spectrum given by (14). Then the intensity of the radiated sound becomes

$$I \simeq \tfrac{3}{32} 2^{\frac{1}{2}} \frac{\rho_0 \tilde{u}^4 U_0^4 D^4}{c^5 R^2 l^2} \frac{1}{k_0 l} \frac{U_0 T_\epsilon}{\langle \epsilon^2 \rangle^{\frac{1}{2}} x_0} \left(1 + \frac{\omega_0^2 \langle \epsilon^2 \rangle l^2}{U_0^2}\right)^2 \mathscr{D}^{-5} \exp\left(-\frac{U_0^2 T_\epsilon^2 (1 - M\cos\theta)^2}{4 x_0^2 \langle \epsilon^2 \rangle \mathscr{D}^2}\right)$$
$$\times \left[1 + \frac{1}{2} \frac{U_0^4 T_\epsilon^4}{\langle \epsilon^2 \rangle^2 x_0^4} \frac{1}{k_0^2 l^2} \left(1 + \frac{\omega_0^2 \langle \epsilon^2 \rangle l^2}{U_0^2}\right) \mathscr{D}^{-2} + \frac{1}{48} \frac{U_0^8 T_\epsilon^8}{\langle \epsilon^2 \rangle^4 x_0^8} \frac{1}{k_0^4 l^4} \left(1 + \frac{\omega_0^2 \langle \epsilon^2 \rangle l^2}{U_0^2}\right)^2 \mathscr{D}^{-4}\right], \tag{17}$$

where $\mathscr{D}$ is the generalized Doppler factor

$$\mathscr{D} = \left[(1 - M\cos\theta)^2 + \frac{1}{2} \frac{U_0^2 T_\epsilon^2}{\langle \epsilon^2 \rangle x_0^2} \frac{U_0^2}{\omega_0^2 l^2} \left(1 + \frac{\omega_0^2 \langle \epsilon^2 \rangle l^2}{U_0^2}\right)\right]^{\frac{1}{2}}.$$

4. Discussion of the wave model

Crow (1972) claimed that a jet could act as an amplifier of an internal tone and that a gain of 30 dB was possible. His model of the source structure suggests a reason for such an amplification, since forcing the jet with a tone amplifies the corresponding instability wave, which then radiates more sound; if randomness in the phase of the wave is negligible, this sound is radiated only at one discrete frequency, the frequency of the tone. It can be seen from (10) that more sound is radiated as the instability wave is amplified. For there is an increase in $\tilde{u}$, the maximum amplitude that can be attained before saturation limits the growth of the velocity fluctuations, and this maximum amplitude is reached nearer the nozzle exit, where the shear layer is more unstable, the change from growth to decay is more abrupt, and so l is smaller (see figure 2). Such an amplification of an internal tone, however, will occur only when randomness in the phase is negligible, and we do not believe that it is normally negligible either in model jets or in full-scale engines. We expect the randomness to be sufficient to ensure that the radiated noise is broad-band.

We examine now the consequences of the hypothesis that the large-scale structure forms the dominant source of jet noise (i.e. that all the noise arises from the 'breaking' of instability waves). For a 'quiet' jet this might not be true since there could be an important contribution from the background turbulence (source location techniques indicated that the noise sources are distributed throughout the jet). But, when the jet is forced, Moore found that all the extra broad-band noise comes from the same place (about three diameters downstream of the nozzle exit, just where the waves break). This suggests that it is the large-scale structure itself, and not an increase in the background turbulence, which makes the extra broad-band noise.

For a 'quiet' jet the forcing by random fluctuations in the boundary layer at the nozzle exit is broad-band, so the most unstable waves grow fastest and saturate nearest the nozzle. We expect the breaking of these waves to make the most noise since the change from growth to decay will not be so abrupt for any other wave (the velocity profile will be more stable and there will be more background turbulence to 'dissipate' the wave's energy and break up its structure). Our model predicts that increased forcing at the frequency of one of the most unstable waves will make the jet noisier. It increases the radiated broad-band noise, for the maximum amplitude of the instability wave is increased and the change from growth to decay is more abrupt. Also, since the wave breaks nearer the nozzle exit, x_0 is reduced and the spectrum of the radiated sound becomes narrower with its peak increasing in amplitude (see figure 2).

Strong enough forcing at a different frequency amplifies the wave at that frequency and can cause it to saturate first. At lower frequencies, because the growth of the waves is so slow, the level of forcing required for this to happen is very high and is unlikely to be present; consequently forcing at low frequencies changes the radiated sound little. But at higher frequencies the level required is quite moderate (see figure 1), so it is possible for a high frequency wave to be the one that breaks nearest the nozzle. We believe that in these circumstances the broad-band noise might be *reduced*. We have already argued that the waves that break nearest the nozzle are the noisiest; because it breaks first, the high frequency wave might succeed in draining the energy from other waves, in particular from the 'most unstable waves', or in destroying their potency as noise sources. Then, because the high frequency waves are relatively quiet when they break (their growth and decay is not very abrupt and the maximum amplitude they can attain is fairly small; see figure 1), there could be a reduction in the radiated broad-band noise.

These predictions agree with experimental results of Bechert & Pfizenmaier (1975) and Moore (1977). They found that forcing a jet by a pure tone can indeed increase the radiated broad-band noise. Moore also showed that forcing at high frequencies could reduce the noise, but only when the boundary layer inside the nozzle was thick. For the forced jet the spectrum of the radiated sound was more peaked than for the unforced jet and the peak frequency varied with the excitation frequency and not with the flow speed, but, in other respects and particularly at high frequencies, the spectrum predicted by our model does not agree well with experimental results. Part of the difference arises, as does the difficulty we have encountered in making order-of-magnitude estimates of the amplitude of the radiated sound, because of the *exponential* reduction in the sound intensity when the growth and decay of the instability waves is slow on the wavelength scale. Estimates from the experiments of Crow & Champagne (see figure 2) suggest that the factor $\frac{1}{2}k_0^2 l^2$ could be as large as 100, and then the exponential reduction would be very large indeed. But it is quite possible that the reduction should not be exponential. If, for example, the growth and decay is exponential rather than Gaussian, varying as

$$\cos\{\omega_0 t - \omega_0(x+x_0)/U_0\}\exp\{-|x|/l\}, \tag{18}$$

we find that the power spectrum of the radiated sound is given by

$$I(\omega) = \frac{\rho_0 \tilde{u}^4 D^4 \omega^4 l^2}{8\pi c^5 R^2}\{1 + k^2 l^2 (1 - M\cos\theta)^2\}^{-2}\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\} \tag{19}$$

instead of (10) for the case when randomness is negligible. The reduction is then algebraic. Clearly the radiated sound is too sensitive to the details of the amplitude function that models the saturation of the waves.

5. A vortex-pairing model

So far we have only modelled randomness in the phase velocity and ignored any variations that occur in the position of saturation. Random fluctuations in the amplitude of the wave at the nozzle exit must give rise to some axial drift in the position at which the eddy development is most rapid, and to investigate the noisiness of this feature we employ a simple model of the abrupt coalescence of eddies in which wave crests are suddenly 'lost'. We assume that the fluctuations in the jet vary sinusoidally along the jet axis and convect downstream without modification at a constant subsonic velocity to the position where pairing occurs. The sinusoidal pattern then changes abruptly (the wavelength and the amplitude are doubled), and this new pattern convects downstream at the same speed without developing further. The fluctuations are therefore proportional to

$$\left.\begin{array}{ll} \cos k_0(x-U_0 t) & \text{for} \quad x < 0, \\ 2\cos\frac{1}{2}k_0(x-U_0 t) & \text{for} \quad x > 0, \end{array}\right\} \tag{20}$$

where U_0 is the convection velocity and k_0 the wavenumber of the eddy pattern before coalescence.

We restrict attention here to subsonic convection velocities. Of course any flow pattern is silent if it convects without modification at a constant subsonic velocity, so any sound is radiated only as a result of pattern development. It might be thought that details of that development are important, but we have checked that they modify the radiated sound only slightly if the pairing occurs sufficiently rapidly. (The only detail that is essential is the quadrupole nature of the source, and this we have incorporated in our model.) For example, there is little change in the radiated sound if the fluctuations are proportional to

$$\left.\begin{aligned} \cos k_0(x-U_0 t)\,(1-e^{x/l}) \quad &\text{for} \quad x<0,\\ 2\cos \tfrac{1}{2}k_0(x-U_0 t)\,(1-e^{-x/l}) \quad &\text{for} \quad x>0,\end{aligned}\right\} \tag{21}$$

provided only that $k_0 l \ll 1$. This is because the Fourier decomposition of the flow pattern is changed only at wavenumbers higher than those well coupled to the sound field. Both experiments and numerical modellings show that pairings occur abruptly, in a distance comparable to the eddy separation, so we believe our model will give reasonable estimates of the sound radiated at low and medium frequencies. But we cannot expect it to give good estimates at frequencies much higher than the passing frequency of the eddies $\omega_0 = k_0 U_0$, since then the details of the coalescence will be important. To incorporate in our model the variation in the position of coalescence, we assume that the eddy structure changes no longer always at $x = 0$ but at $x = g(t)$, where g is a normally distributed random function with zero mean; the fluctuations are proportional to

$$\left.\begin{aligned} \cos k_0(x-U_0 t) \quad &\text{for} \quad x<g(t),\\ 2\cos \tfrac{1}{2}k_0(x-U_0 t) \quad &\text{for} \quad x>g(t).\end{aligned}\right\} \tag{22}$$

A composite picture of the jet's development obtained from traces of the axial distribution of these fluctuations at successive times is illustrated in figure 8. (In this figure the fluctuations have been made continuous at $x = g(t)$.) The general features of the jet's development are similar to those obtained by Acton in her numerical modelling of the forced jet (figure 6).

6. The sound radiated by vortices pairing

We obtain an estimate of the radiated sound by solving for the quadrupole-driven density field, given by

$$\frac{\partial^2\rho}{\partial t^2} - c^2\frac{\partial^2\rho}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i\,\partial x_j},$$

assuming Lighthill's stress tensor T_{ij} to be a line source with the structure of the fluctuations in the jet.

We first determine the importance of the details of the eddy coalescence by neglecting variations in the pairing position and estimating Lighthill's stress tensor as

$$T_{ij} = \left\{\begin{aligned} &\rho_0\tilde{u}^2D^2\delta_{ij}\,\delta(y)\,\delta(z)\cos[k_0(x-U_0t)]\,(1-e^{x/l}) \quad \text{for} \quad x<0,\\ &2\rho_0\tilde{u}^2D^2\delta_{ij}\,\delta(y)\,\delta(z)\cos[\tfrac{1}{2}k_0(x-U_0t)]\,(1-e^{-x/l}) \quad \text{for} \quad x>0.\end{aligned}\right\} \tag{23}$$

Here D is the jet diameter, c is the speed of sound, ρ_0 is the ambient density and l is a parameter that we vary to model differences in the details of the eddy pairing [*cf.* (21)]. Then

$$\begin{aligned}\lim_{R\to\infty} p(\mathbf{x},t) &= \frac{\rho_0\tilde{u}^2D^2}{4\pi c^2 R}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}[\cos\{k_0\xi(1-M\cos\theta)-k_0U_0(t-R/c)\}\,(1-e^{\xi/l})\,H(-\xi)\\ &\quad +2\cos\{\tfrac{1}{2}k_0\xi(1-M\cos\theta)-\tfrac{1}{2}k_0U_0(t-R/c)\}\,(1-e^{-\xi/l})\,H(\xi)]\,d\xi\\ &\simeq \frac{\rho_0\tilde{u}^2U_0^2k_0D^2}{4\pi c^2R(1-M\cos\theta)}[\sin\{k_0U_0(t-R/c)\}-\sin\{\tfrac{1}{2}k_0U_0(t-R/c)\}],\end{aligned} \tag{24}$$

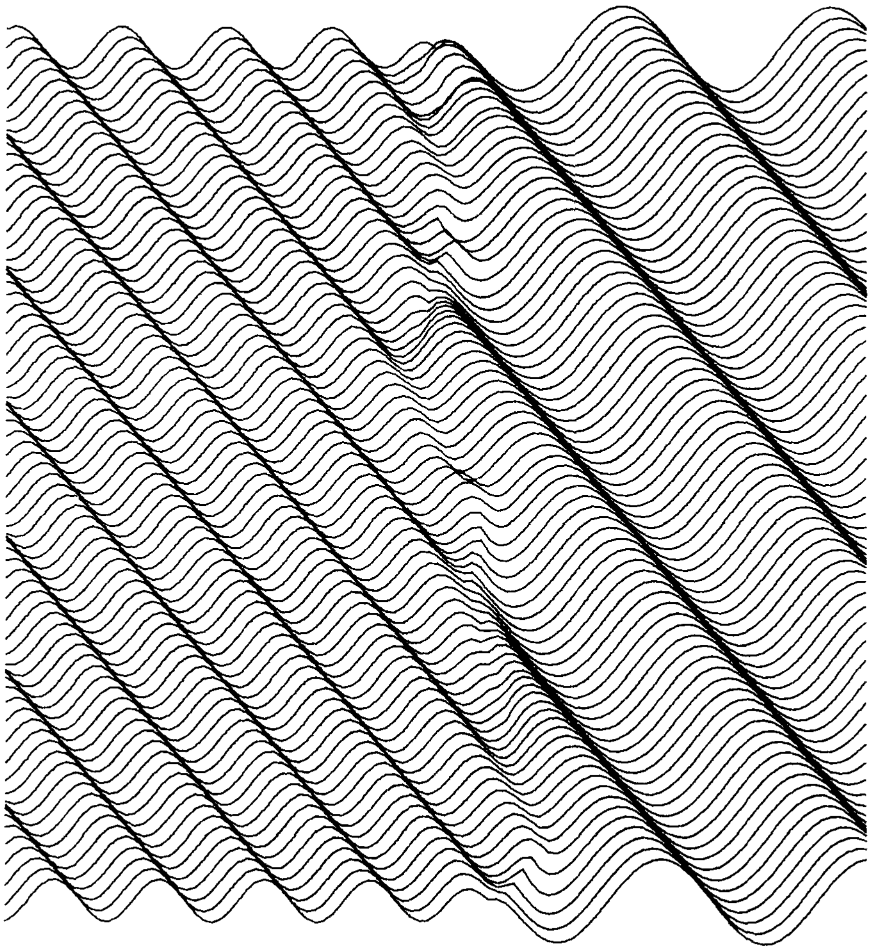


FIGURE 8. A composite picture of the jet's development. Traces at successive times of the axial distribution of the fluctuations:

$$\cos [k_0(x-U_0 t)]\{1-\exp [(x-g(t))/l]\} \quad \text{for} \quad x < g(t),$$
$$2 \cos [\tfrac{1}{2}k_0(x-U_0 t)]\{1-\exp [(g(t)-x)/l]\} \quad \text{for} \quad x > g(t).$$

with a relative error of order $k_0 l(1-M\cos\theta)$. Here $R = |x|$, $\cos\theta = x/R$, $M = U_0/c$ and H is the Heaviside function. The radiated sound depends little on the value of l provided that $k_0 l \ll 1$; so we believe that to estimate the radiated sound we do not require a detailed knowledge of the motions of the eddies as they coalesce, provided that such coalescence occurs in a distance very small compared with the eddy separation.

The position where eddies pair is observed in experiments to vary over a distance which is comparable to the eddy separation even when the jet is forced, so it is important that we include this variation in our model. We therefore assume [cf. (22)] that Lighthill's stress tensor is given by

$$T_{ij} = \begin{cases} \rho_0 \tilde{u}^2 D^2 \delta_{ij}\,\delta(y)\,\delta(z)\cos k_0(x-U_0 t) & \text{for} \quad x < g(t), \\ 2\rho_0 \tilde{u}^2 D^2 \delta_{ij}\,\delta(y)\,\delta(z)\cos \tfrac{1}{2}k_0(x-U_0 t) & \text{for} \quad x > g(t). \end{cases} \tag{25}$$

Then

$$\lim_{R\to\infty} p(\mathbf{x},t) = \frac{\rho_0 \tilde{u}^2 D^2}{4\pi c^2 R}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} [\cos\{k_0\xi(1-M\cos\theta)-k_0 U_0(t-R/c)\}$$
$$\times H\{g(t-R/c+\xi\cos\theta/c)-\xi\}$$
$$+2\cos\{\tfrac{1}{2}k_0\xi(1-M\cos\theta)-\tfrac{1}{2}k_0 U_0(t-R/c)\}H\{\xi-g(t-R/c+\xi\cos\theta/c)\}]\,d\xi$$
$$\simeq \frac{\rho_0 \tilde{u}^2 D^2}{4\pi c^2 R}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} [\cos\{k_0\xi - k_0 U_0(t-R/c)\}H\{g(t-R/c)-\xi\}$$
$$+2\cos\{\tfrac{1}{2}k_0\xi-\tfrac{1}{2}k_0 U_0(t-R/c)\}H\{\xi-g(t-R/c)\}]\,d\xi, \tag{26}$$

if we neglect differences in the retarded time across the source region. This introduces relative errors of order $M\cos\theta$ and of order $M\cos\theta k_0\langle g^2\rangle U_0 T_g$, where $\langle g^2\rangle^{\frac{1}{2}}$ is the r.m.s. value and T_g the correlation time for g. So

$$\lim_{R\to\infty} p(\mathbf{x},t) \simeq \frac{\rho_0 \tilde{u}^2 D^2}{4\pi c^2 R k_0}\frac{\partial^2}{\partial t^2}[\sin\{k_0 g(t-R/c)-k_0 U_0(t-R/c)\}$$
$$-4\sin\{\tfrac{1}{2}k_0 g(t-R/c)-\tfrac{1}{2}k_0 U_0(t-R/c)\}] \tag{27}$$

and

$$I(\omega) \simeq \frac{\rho_0 \tilde{u}^4\omega^4 D^4}{16\pi^2 c^5 R^2 k_0^2}\int_{-\infty}^{\infty} \cos\omega\tau\langle[\sin\{k_0 g(t-R/c)-k_0 U_0(t-R/c)\}$$
$$-4\sin\{\tfrac{1}{2}k_0 g(t-R/c)-\tfrac{1}{2}k_0 U_0(t-R/c)\}]$$
$$\times[\sin\{k_0 g(t+\tau+R/c)-k_0 U_0(t+\tau-R/c)\}$$
$$-4\sin\{\tfrac{1}{2}k_0 g(t+\tau-R/c)-\tfrac{1}{2}k_0 U_0(t+\tau-R/c)\}]\rangle\,d\tau$$
$$\simeq \frac{\rho_0 \tilde{u}^4\omega^4 D^4}{32\pi^2 c^5 R^2 k_0^2}\,\mathrm{Re}\int_{-\infty}^{\infty} \cos\omega\tau\,\langle\exp[ik_0\{g(t-R/c)-g(t+\tau-R/c)\}+ik_0 U_0\tau]$$
$$+16\exp[\tfrac{1}{2}ik_0\{g(t-R/c)-g(t+\tau-R/c)\}+\tfrac{1}{2}ik_0 U_0\tau]\rangle\,d\tau$$
$$\simeq \frac{\rho_0 \tilde{u}^4\omega^4 D^4}{32\pi^2 c^5 R^2 k_0^2}\int_{-\infty}^{\infty} \cos\omega\tau\,(\exp[-k_0^2\langle g^2\rangle\{1-R_g(\tau)\}]\cos\omega_0\tau$$
$$+16\exp[-\tfrac{1}{4}k_0^2\langle g^2\rangle\{1-R_g(\tau)\}]\cos\tfrac{1}{2}\omega_0\tau)\,d\tau, \tag{28}$$

where Re indicates the real part, R_g is the autocorrelation coefficient for g, and $\omega_0 = k_0 U_0$. In reaching this result we have again used (8).

If the variation in the position of pairing is small, i.e. $\langle g^2\rangle k_0^2 \ll 1$, the integrand in (28) may be expanded as a power series in $\langle g^2\rangle k_0^2$. The spectrum of the radiated sound is then given to lowest order by

$$I(\omega) \simeq \frac{\rho_0 \tilde{u}^4 U_0^4 k_0^2 D^4}{32\pi c^5 R^2}\{\delta(\omega+\omega_0)+\delta(\omega-\omega_0)+\delta(\omega+\tfrac{1}{2}\omega_0)+\delta(\omega-\tfrac{1}{2}\omega_0)\}; \tag{29}$$

sound is radiated only at certain discrete frequencies. But when the variation in the position of pairing is large, i.e. $\langle g^2\rangle k_0^2 \gg 1$, the τ integration in (28) may be performed by the method of steepest descents, the integral being dominated by the contribution from the region near $\tau = 0$. Since $d^2R_g(\tau)/d\tau^2 = -2/T_g^2$ when $\tau = 0$, we find that

$$I(\omega) \simeq \frac{33}{32}\frac{\rho_0 \tilde{u}^4\omega^4 D^4}{\pi^{\frac{3}{2}} c^5 R^2 k_0^2}\frac{T_g}{k_0\langle g^2\rangle^{\frac{1}{2}}}. \tag{30}$$

This expression is not valid at very high frequencies, when variations in the phase factor $\cos\omega\tau$ are important. Then we approximate $R_g(\tau)$ by $1-\tau^2/T_g^2$ in (28) to obtain

$$\begin{aligned}I(\omega) &\simeq \frac{\rho_0\tilde{u}^4\omega^4D^4}{32\pi^2c^5R^2k_0^2}\int_{-\infty}^{\infty}\{\exp(-k_0^2\langle g^2\rangle\tau^2/T_\epsilon^2)\cos\omega\tau\cos\omega_0\tau \\ &\quad + 16\exp(-\tfrac{1}{4}k_0^2\langle g^2\rangle\tau^2/T_\epsilon^2)\cos\omega\tau\cos\tfrac{1}{2}\omega_0\tau\}\,d\tau \\ &\simeq \frac{\rho_0\tilde{u}^4\omega^4D^4}{64\pi^{\frac{3}{2}}c^5R^2k_0^2}\frac{T_g}{k_0\langle g^2\rangle^{\frac{1}{2}}}\left\{\exp\left(-\frac{(\omega+\omega_0)^2T_g^2}{4k_0^2\langle g^2\rangle}\right)+\exp\left(-\frac{(\omega-\omega_0)^2T_g^2}{4k_0^2\langle g^2\rangle}\right)\right. \\ &\quad \left. + 32\exp\left(-\frac{(\omega+\frac{1}{2}\omega_0)^2T_g^2}{k_0^2\langle g^2\rangle}\right)+32\exp\left(-\frac{(\omega-\frac{1}{2}\omega_0)^2T_g^2}{k_0^2\langle g^2\rangle}\right)\right\}. \end{aligned} \tag{31}$$

The radiated sound is broad-band. The width of the spectrum is proportional to $k_0\langle g^2\rangle^{\frac{1}{2}}$, the randomness in the phase of the wave at the position of coalescence, and inversely proportional to T_g, the correlation time of the randomness.

The overall sound intensity I is obtained by integrating the power spectrum over all frequencies. When the variation in the position of pairing is small, we obtain from (29)

$$I \simeq \rho_0\tilde{u}^4U_0^4k_0^2D^4/8\pi c^5R^2, \tag{32}$$

with a relative error of order $\langle g^2\rangle k_0^2$. When the variation is large, we obtain from (31)

$$I \simeq \frac{\rho_0\tilde{u}^4U_0^4k_0^2D^4}{8\pi c^5R^2}\left(1+\frac{12\langle g^2\rangle}{T_g^2U_0^2}+\frac{12\langle g^2\rangle^2}{T_g^4U_0^4}\right). \tag{33}$$

7. Discussion of the vortex-pairing model

Laufer *et al.* (1973) and Winant & Browand (1974) have suggested that the pairing of vortex rings is the mechanism primarily responsible for the generation of jet noise. We now examine how effective this hypothesis is in explaining the characteristics of the noise from excited and unexcited jets.

Our vortex-pairing model does not describe very well the large-scale structure of an unforced jet since the strengths and spacings of the eddies vary a lot and pairings occur at random all along the jet axis. Noise is then generated everywhere, higher frequencies coming mainly from near the nozzle and lower frequencies from further downstream. But our model describes much better the large-scale structure of a forced jet. Forcing modifies the jet structure, inducing those unstable waves at the forcing frequency to grow and form the dominant jet eddies. The spacings of the eddies are more regular so the position of the first pairing always occurs near the same place, though it still varies over a distance comparable to the eddy separation. Since the strengths of the eddies are increased, the pairing process is noisier; forcing a subsonic jet can increase the broad-band noise. The field shapes of the radiated sound would be the same with and without forcing. But, unlike the noise of an unexcited jet, we expect all the additional noise of a forced jet to originate from the same place regardless of its frequency, and the spectrum of this additional broad-band noise to be more peaked with the peak frequency varying with the forcing frequency rather than with the jet velocity.

All these features are consistent with experimental results. However our model does not predict very well the noise at high frequencies, since this noise depends strongly on the details of the eddy coalescence. Nor does it predict that high frequency forcing can reduce the broad-band noise. Moore found that this reduction occurs only if the boundary layer inside the nozzle is thick; perhaps it is too thick to allow instability waves at the forcing frequency to grow to maturity, although these waves are still able to inhibit the growth of others.

We now determine the magnitude of the radiated sound predicted by our model. The coherent velocity fluctuations grow to an amplitude equal to about 7 % of the mean velocity (Moore 1977), so we assume $\tilde{u}/U_J \simeq 0{\cdot}07$, where U_J is the jet velocity; the ratio U_0/U_J of the convection velocity to the jet velocity is typically about 0·6. From the work of Petersen *et al.* (1974) we estimate that the position where pairing occurs varies over a distance comparable to the vortex separation $k_0\langle g^2\rangle^{\frac{1}{2}} \simeq 2\pi$; we do not expect the position of one pairing greatly to affect the position of any other, so we assume that the random function g is correlated typically for the time it takes two eddies to coalesce, $T_g\,\omega_0 \simeq 4\pi$. So $T_g^2U_0^2/\langle g^2\rangle \simeq 4$. We use these estimates to predict the sound radiated fifty diameters away at 90° to the jet axis when a jet with a Mach number U_J/c of 0·3 is forced at a Strouhal number fD/U_J of 0·3 ($\omega_0 = 2\pi f$). Our vortex-pairing model predicts [equation (33)] that the overall sound intensity is 72 dB while for this condition and at this location Moore measured the overall sound intensity to be 74 dB. Of course it is largely coincidental that the predictions of our fairly crude model agree so well with experiment.

8. The Mach angle

The vortex-pairing model we have described is appropriate only for jets with subsonic convection velocities. It predicts, as Bechert & Pfizenmaier (1975) and Moore (1977) have found, that forcing the jet augments the noise over a broad frequency band and not just that at the forcing frequency. Crow (1972), however, claimed that jets could amplify internal tones, and that the amplification was particularly efficient at the Mach angle. In order to see whether amplification of internal tones is to be expected near the Mach angle, we return to our model of the large-scale structure of the jet in terms of instability waves; at the Mach angle this model is not too sensitive to the details of the waves' growth and decay.

We assume that random fluctuations in the strength of the wave at the nozzle exit cause the position of saturation to vary, and estimate Lighthill's stress tensor as

$$T_{ij} = \rho_0\tilde{u}^2D^2\delta_{ij}\,\delta(y)\,\delta(z)\cos\{\omega_0 t-\omega_0(x+x_0)/U_0\}\exp\{-(x-g(t-x/U_0))^2/l^2\}. \quad (34)$$

We again represent the jet as a line quadrupole source on the jet axis. The source strength varies sinusoidally along the jet, its amplitude growing, saturating and then decaying, with the position where the peak amplitude is attained varying randomly.

The far-field fluctuating pressure is

$$\lim_{R\to\infty} p(\mathbf{x},t) = \frac{\rho_0\tilde{u}^2D^2}{4\pi c^2R}\frac{\partial^2}{\partial t^2}\left\{\int_{-\infty}^{\infty}\cos\{\omega_0(t-R/c)+\omega_0\,\xi\cos\theta/c-\omega_0(\xi+x_0)/U_0\}\right.$$
$$\left.\times\exp\{-[\xi-g(t-R/c-\xi(1-M\cos\theta)/U_0)]^2/l^2\}\,d\xi\right\}. \quad (35)$$

At the Mach angle this simplifies to

$$\lim_{R\to\infty} p(\mathbf{x},t) = \frac{\rho_0\tilde{u}^2D^2}{4\pi c^2R}\frac{\partial^2}{\partial t^2}\left\{\int_{-\infty}^{\infty}\cos\{\omega_0(t-R/c)-\omega_0x_0/U_0\}\exp\{-(\xi-g(t-R/c))^2/l^2\}\,d\xi\right\}$$
$$= -\frac{\rho_0\tilde{u}^2D^2\,l\omega_0^2}{4\pi^{\frac{1}{2}}c^2R}\cos\{\omega_0(t-R/c)-\omega_0x_0/U_0\}. \quad (36)$$

Mach-wave sound is evidently radiated only at one frequency regardless of any randomness in the position of saturation. The spectrum of the radiated sound is

$$I(\omega) = (\rho_0 \tilde{u}^4 \omega_0^4 D^4 l^2 / 32c^5 R^2)\{\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\}, \tag{37}$$

and the overall sound intensity is

$$I = \rho_0 \tilde{u}^4 \omega_0^4 D^4 l^2 / 16c^5 R^2. \tag{38}$$

All the signals emitted by a wave, as it grows, saturates and decays, arrive simultaneously at the Mach angle; so it makes no difference to the observer where the wave reaches its maximum amplitude, and randomness in the position of saturation cannot broaden the spectrum of the radiated sound. We consequently find Crow's results quite plausible. They are not inconsistent with the more recent lower-speed surveys of Bechert & Pfizenmaier and Moore.

9. Conclusions

We have proposed two models of the acoustic sources in a turbulent jet. In the first we assume that they have a structure similar to instability waves which initially grow on an unstable shear layer, but then saturate and eventually decay. The abruptness of the change from growth to decay determines the magnitudes of the radiated sound while the randomness in the phase velocity of the waves determines its spectrum; broad-band noise is radiated if the randomness is large.

Forcing the jet at one frequency by, for example, an internal noise source amplifies the corresponding wave and we show that this increases the radiated broad-band noise. Such an increase has been observed experimentally, and our model is able to predict accurately many of its details. In particular we predict that the field shapes and spectra for the excited and unexcited jet are similar, though the spectrum for the excited jet is more peaked and the peak frequency depends on the forcing frequency and not on the mean flow speed. Also we predict that all the extra noise, regardless of its frequency, originates from the same place when a jet is forced. Finally we argue that jet noise could be dominated by the noise of the wave that 'breaks' nearest the nozzle, so that high frequency forcing could reduce the broad-band noise. But our model does not predict very well the noise radiated at high frequencies; and the magnitude of the radiated sound is difficult to estimate because it is too sensitive to the details of the change from growth to decay of the waves.

Our second model examines the noise radiated when two eddies coalesce. Since the coalescence occurs abruptly, we believe that its details are important only at high frequencies and need not be incorporated in our model. The magnitude of the radiated sound is then determined by the eddy strength, and the spectrum by the randomness in the position of pairing. Even when the jet structure is made more regular by upstream forcing, the position where pairing occurs varies over a distance comparable to the eddy separation, and then our model predicts that the radiated sound is broadband.

It has been suggested that the pairing of eddies is the mechanism primarily responsible for the production of jet noise, and our results support this hypothesis. Our vortex-pairing model predicts the characteristics of excited jets better than our wave model does (though, as we should expect, the vortex-pairing model is still not very good at high frequencies). We find that with the vortex-pairing model estimates of the radiated sound are not so sensitive to the details of the flow and in fact agree well with the values found experimentally. The model does not predict a reduction in the broadband noise with high frequency forcing, but this, we suggest, might occur when the boundary layer inside the nozzle is thick because waves at the forcing frequency cannot grow on a thick shear layer though they are still able to inhibit the growth of other waves. For jets with convection speeds faster than the ambient speed of sound, we predict that the noise radiated at the Mach angle is narrow-band, so there the jet could act as an amplifier of an internal tone.

The experimental support for many of the predictions of our model suggests that the real sources in a jet might have a simple structure similar to that which we describe.

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